Topology Final Test
(BMath-2nd year, 2025)

Instructions: Total time 3 Hours. Solve as many problems as you like, for a maximum score of 60. Use terminologies, notations and results as covered in the course, no need to prove such results. If you wish to use a problem given in an assignment or homework or any other such, please supply its full solution.

1. Let $r > 0$ and $p \in \mathbb{R}^n$ be arbitrary. Prove that $\mathbb{R}^n$ is homeomorphic to the open ball $B(p, r)$ with center p and radius r in $\mathbb{R}^n$ by giving an explicit homeomorphism. (5)

2. Let X be any of the following topological spaces : (a) Klein bottle (b) projective plane (c) torus (d) Möbius band. Prove that the complement of any finite set in X is path connected. (10)

3. Let X denote the Möbius band with its boundary circle removed. Prove that the one point compactification of X is homeomorphic to $\mathbb{R}P^2$. (10)

4. Prove that any group automorphism of S^1 which is also a homeomorphism is either the identity map $z \mapsto z$ for all $z \in S^1$ or the inversion $z \mapsto z^{-1}$, for all $z \in S^1$. (10)

5. Let G denote the group of self-homeomorphisms of $\mathbb{R}^2$ generated by the maps $\tau((x, y)) = (x + 1, y)$ and $\eta((x, y)) = (-x, y + 1)$. Prove that the quotient space $\mathbb{R}^2/G$ is homeomorphic to the Klein bottle. (15)

6. Let X be a compact Hausdorff space and $C(X)$ denote the cone over X. Let $Y = X \times [0, 1)$ with product topology. Prove that the one point compactification of Y is homeomorphic to $C(X)$. (10)

7. Prove that the metric space $(\mathcal{C}([0, 1]), \sup)$ of all continuous real valued functions on $[0, 1]$, is not locally compact. (Hint: Consider the functions $f_n(x) = nx,\ 0 \leq x \leq 1/n,\ \ f(x) = 1,\ 1/n \leq x \leq 1.$) (10)

8. Let $p : \mathbb{R}^2 \to [0, \infty)$ be given by $p((x, y)) = x^2 + y^2$. Is p a quotient map? Explain. (5)

9. Let X and Y be topological spaces with X compact. Prove that the projection map $\pi : X \times Y \to Y$ is a closed map. (10)